Dual parameterisation for linear nonminimum phase plants

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Abstract: Dual parameterisation has recently been presented for the class of all stabilising controllers for linear minimum phase rational plants. It is shown that the result can be extended to the parameterisation of all stabilising controllers for linear nonminimum phase rational plants. The parameterisation for nominal plants is a special case of the new parameterisation. Numerical examples are provided to illustrate the new parameterisation.

1 Introduction

Given a linear time-invariant plant, a basic design problem is to find a linear time-invariant controller that will stabilise the plant and meet various design objectives. Thus, the problem of stabilisation is always an active research area in the design of control systems. Many results relating to stability analysis have been presented, of which the most famous is probably Youla parameterisation [1, 2], which has been widely used for designing control systems. The technique enables one to obtain all stabilising controllers for a closed loop system.

Recently, the authors of [3] presented a simple parameterisation for the class of all stabilising controllers for linear minimum phase rational plants. This result is the dual of the well-known Youla parameterisation for the class of all stabilising controllers for linear stable plants. In [4] it is shown that such parameterisation can also be used for stable nonminimum phase rational plants. In this paper, a modified version will be developed which extends the parameterisation for [3] for linear nonminimum phase rational plants, or equivalently, all linear rational plants.

2 Main results

Consider the unity feedback control system shown in Fig. 1, where $C(s)$ is the controller and $G(s)$ is the plant. Supposing that the plant is stable and the model is exact, the controller that makes the closed loop system internally stable can be parameterised as [1, 2, 5]

$$C(s) = \frac{Q(s)}{1 - G(s)Q(s)}$$

(1)

where $Q(s)$ is a stable transfer function. This is the Youla parameterisation for stable plants. The sensitivity function, i.e. the transfer function from the external disturbance $d(s)$ to the plant output $y(s)$ is given by

$$S(s) = \frac{1}{1 + G(s)C(s)} = 1 - G(s)Q(s)$$

(2)

the complementary sensitivity function, i.e. the transfer function from the setpoint input $r(s)$ to the plant output $y(s)$ is given by

$$T(s) = \frac{G(s)C(s)}{1 + G(s)C(s)} = G(s)Q(s)$$

(3)

Suppose that $G(s)$ is minimum phase, but not necessarily stable, the closed loop system shown in Fig. 1 is internally stable if and only if $C(s)$ is parameterised as [3]

$$C(s) = Q'(s)^{-1} - G(s)^{-1}$$

(4)

where $Q'(s)$ is a stable nonzero transfer function. It follows that

$$S(s) = \frac{Q'(s)}{G(s)}$$

(5)

and

$$T(s) = 1 - \frac{Q'(s)}{G(s)}$$

(6)

For stable and minimum phase systems this reveals a duality between the two parameterisations.

Let us now consider the nonminimum phase plant $G(s)$. A general form for the nonminimum phase plant is as follows

$$G(S) = K \frac{N_+(s)N_-(s)}{M_+(s)M_-(s)}$$

(7)

where $K$ is the gain, $N_+(s)$, $N_-(s)$, $M_+(s)$, and $M_-(s)$ are polynomials with the constant term being 1. The subscript +

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Fig. 1 Feedback control system
denotes that all roots are in the right half plane and \(M(s)\) denotes that all roots are in the left half plane. Because the plant is always proper, \(\deg\{N_r(s)\} + \deg\{N_e(s)\} < \deg\{M_e(s)\} + \deg\{M_e(s)\}\). Suppose that the plant has \(r\) poles in the right half plane and \(s_j\) \((i = 1, 2, \ldots, r)\) is of multiplicity \(l_i\).

**Theorem 1:** The closed loop system shown in Fig. 1 is internally stable if and only if \(C(s)\) is parameterised as

\[
C(s) = Q(s)^{-1} - G(s)^{-1}
\]

where \(Q(s)\) is a stable nonzero transfer function of the form

\[
Q(s) = G(s)(1 - Q'(s)N_r(s))
\]

and \(Q'(s)\) is a stable nonzero transfer function satifying

\[
\lim_{{s \to s_i}} \frac{d^k}{ds^k}(1 - Q'(s)N_r(s)) = 0, \quad 0 \leq k < l_i
\]

**Proof:** From the definition it is known that the necessary and sufficient conditions guaranteeing the internal stability of the closed loop system are that both \(G(s)/(1 + G(s)C(s))\) and \(C(s)/(1 + G(s)C(s))\) are stable. It is easy to verify that

\[
Q(s) = \frac{G(s) - Q(s)}{G(s)^2} = \frac{KM_e(s)M_e(s)N_r(s)N_e(s) - M_e(s)^2M_e(s)^2Q'(s)}{K^2N_r(s)^2N_e(s)^2}
\]

\[(12)\]

If \(Q(s)\) is stable, all right half plane poles should be cancelled by a stable \(Q'(s)\). Hence, \(Q'(s)\) must be of the form

\[
Q'(s) = K\frac{N_e(s)N_r(s)}{M_e(s)M_e(s)}(1 - Q'(s)N_r(s))
\]

\(Q'(s)\) is a stable transfer function. It should be selected such that \(Q'(s)\) is stable, i.e. \(1 - Q'(s)N_r(s)\) should have zeros at the poles of \(M_e(s)\). This is equivalent to

\[
\lim_{{s \to s_i}} \frac{d^k}{ds^k}(1 - Q'(s)N_r(s)) = 0, \quad 0 \leq k < l_i
\]

Obviously, the sensitivity and complementary sensitivity function are affine in both \(Q'(s)\) and \(Q''(s)\). We now look into several special cases for which Theorem 1 can be simplified further.

**Corollary 2:** Suppose that the plant is stable (i.e. \(M_e(s) = 1\)). The closed loop system is internally stable if and only if \(C(s)\) is parameterised as

\[
C(s) = Q'(s)^{-1} - G(s)^{-1}
\]

where

\[
Q'(s) = G(s)(1 - Q'(s)N_r(s))
\]

and \(Q'(s)\) is any stable transfer function.

**Proof:** Since \(G(s)\) is stable, \(Q'(s)\) is stable if and only if \(Q''(s)\) is stable. This implies that \(Q'(s)\) can be any stable transfer function.

It is easy to verify that corollary 2 is identical to that of the Youla parameterisation for stable plants.

**Corollary 3:** Suppose that the plant is minimum phase (i.e. \(N_r(s) = 1\)). The closed loop system is internally stable if and only if \(C(s)\) is parameterised as

\[
C(s) = Q'(s)^{-1} - G(s)^{-1}
\]

where \(Q(s)\) is any stable nonzero transfer function.

**Proof:** For minimum phase plants, \(Q'(s)\) stable implies \(Q(s)\) stable. Hence, \(Q(s)\) can be any stable transfer function.

Corollary 3 is just the result of [3]. Hence, the parameterisation of [3] is a special case of the proposed parameterisation.

**Corollary 4:** Suppose that the plant has only one distinct right half plane pole \(s_i\). The closed loop system is internally stable if and only if \(C(s)\) is parameterised as

\[
C(s) = Q'(s)^{-1} - G(s)^{-1}
\]

where \(Q(s)\) is a stable nonzero transfer function of the form

\[
Q'(s) = G(s)(1 - Q'(s)N_r(s))
\]

and \(Q'(s)\) is a stable transfer function of the form

\[
Q''(s) = \frac{1}{N_r(s_i)} + (s - s_i)Q''(s)
\]

Here \(Q''(s)\) is any stable transfer function.

**Proof:** \(Q'(s)\) should satisfy

\[
\lim_{{s \to s_i}} (1 - Q''(s)N_r(s)) = 0
\]

This implies that

\[
Q'(s_i) = \frac{1}{N_r(s_i)}
\]

or equivalently

\[
Q'(s_i) = \frac{1}{N_r(s_i)} + (s - s_i)Q''(s)
\]

\(Q''(s)\) is any stable transfer function.

Assume that the linear nonminimum phase plant is factorised as

\[
G(s) = \frac{N(s)}{M(s)}
\]

\[(19)\]

where \(N(s)\) and \(M(s)\) are coprime stable proper rational transfer functions. Let \(X(s)\) and \(Y(s)\) be stable proper rational transfer functions satisfying

\[
N(s)X(s) + M(s)Y(s) = 1
\]

\[(20)\]

The Youla parameterisation for nonminimum phase plants can be written as

\[
C(s) = \frac{X(s) + M(s)Q(s)}{Y(s) - N(s)Q(s)}
\]

\[(21)\]

where \(Q(s)\) is any stable proper function. Compared with the Youla parameterisation, the main merit of the proposed parameterisation is that it is convenient for some problems and thus provides an alternative.
Example 1: Consider the following open loop unstable plant

\[ G(s) = \frac{s - 2}{(s - 1)(s + 2)} \]

There is only one right half plane pole. By (18) we have

\[ Q''(s) = 2 + (s - 1)Q''(s) \]

where \( Q''(s) \) is any stable transfer function. In this case, one obtains

\[ Q'(s) = \frac{(s - 2)(2 + Q''(s)(s - 2))}{2(s + 2)} \]

and

\[ C(s) = -\frac{(s + 2)(2 + Q''(s)(s - 1))}{2 + Q''(s)(s - 2)} \]

Example 2: Consider an unstable plant with two poles in the right half plane

\[ G(s) = \frac{1}{(s - 1)(s - 2)} \]

The plant is minimum phase, \( N_p(s) = 1 \). According to corollary 3

\[ C(s) = Q'(s)^{-1} - (s - 1)(s - 2) \]

where \( Q'(s) \) is any stable nonzero transfer function.

The proposed parameterisation, as well as the Youla parameterisation, usually results in an improper parameter. The realisation of such a controller is not a problem in the frequency domain, since an improper transfer function can be approximated over any desired frequency range by a proper one. The method has been widely used in modern frequency domain controller design methods (see, for example, [5, 6]).

3 Conclusions

In this paper, we have discussed the parameterisation of linear nonminimum phase rational plants and extended the results of [3]. It has been shown that the parameterisation of minimum phase plants can be used for nonminimum phase plants in a similar form, and the sensitivity and complementary sensitivity function are affine (i.e. linear plus a constant) in both \( \hat{G}(s) \) and \( Q'(s) \). The most important merit of the proposed parameterisation is that it is convenient for some problems and thus provides an alternative. How to use the parameterisation to design the controller is being studied and will be the subject of another paper.

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5 References