Algebraic Solution to H2 Global Optimal Control

Brief statement of the problem

Given a linear plant, analytically design the optimal and suboptimal controller such that

1) The closed-loop system is internally stable;
2) The performance and robustness can be quantitatively tuned.

The performance index is to minimize the 2 norm of the weighting sensitivity function. The obtained solution is the unique optimal solution.

Features

Compared with the classical decoupling control, the proposed method has two important merits:

1) The method can directly deal with plants with imaginary axis poles.
2) The weighting function is unified and different designers can obtain the same controller.
3) The design procedure is optimal and analytic.
4) No state variables and no observer are used.
5) No augmented plant is used and the controller is of lower order.
6) It provides simple tuning rule for quantitative performance and robustness.

Design procedure

1) $G(s)$ is factored into

$$
G(s) = G_A(s)G_{MP}(s)
$$

where

$$
G_A(s) = I - B^* (sI + \overline{A})^{-1} F^{-1} B
$$

(if $z_j$ is a simple RHP zero of $G(s)$,

$$
A = \begin{bmatrix}
  z_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & z_{r_z}
\end{bmatrix},
B = \begin{bmatrix}
  v_1 \\
  \vdots \\
  v_{r_z}
\end{bmatrix},
F = \begin{bmatrix}
  f_{1j} \\
  \vdots \\
  f_{r_z j}
\end{bmatrix},
F_{ij} = \frac{v_i v_j^*}{z_j + z_i}, i,j = 1, 2, \ldots, r_z.
$$

$v_j$ is the direction of the zero $z_j$.}
2) The optimal controller is

\[ Q_{opt}(s) = G_{opt}^{-1}(s)G_{i}^{-1}(0) \]

3) Introduce J(s) to roll the optimal controller off: \( Q(s) = Q_{opt}(s)J(s) \). The unity feedback controller is

\[ C(s) = Q(s)[I - G(s)Q(s)]^{-1} \]

Tuning rule for performance and robustness:
Increase the performance degrees from small to large until the required response is obtained.

**Examples**

**Example 1**  Consider the following plant

\[
G(s) = \frac{1}{(s+1)^2} \begin{bmatrix} (s-1)^2 & (s-1)^2 \\ (s-1)(s-2) & 2(s-1)(s-2) \end{bmatrix}
\]

The optimal IMC controller is

\[
Q(s) = \frac{s+1}{s+2} \begin{bmatrix} 2(s+2) & -(s+1) \\ \lambda_1 s + 1 & \lambda_2 s + 1 \\ -(s+2) & s+1 \\ \lambda_1 s + 1 & \lambda_2 s + 1 \end{bmatrix}
\]

The unity feedback loop controller is

\[
C(s) = \begin{bmatrix} 2(s+1)^3 & -(s+1)^3 \\ s(\lambda_1 s^2 + 2\lambda_1 s + \lambda_2 + 4) & s(\lambda_2 s^2 + 3\lambda_2 s + 2\lambda_2 + 6) \\ -\lambda_1 s^2 + 2\lambda_1 s + \lambda_2 + 4 & -s(\lambda_2 s^3 + 3\lambda_2 s + 2\lambda_2 + 6) \end{bmatrix}
\]

Suppose the design specification is 20% undershoot with the shortest rise time for both the two loops. We can take \( \lambda_1=1.25 \) and \( \lambda_2=1.05 \).
Example 2 Consider the following plant

\[ G(s) = \frac{1}{s+1} \begin{bmatrix} s-1 & s-1 \\ 1 & s-2 \end{bmatrix} \]

The optimal IMC controller is

\[ Q(s) = \frac{1}{5(s+1)} \begin{bmatrix} -(7s+10) & -(s-5) \\ \lambda_2 s + 1 & \lambda_2 s + 1 \\ 4s + 5 & -(3s + 5) \\ \lambda_1 s + 1 & \lambda_2 s + 1 \end{bmatrix} \]

Suppose the design specification is the \( y_2 < 0.3 \) for \( r_1 = 1/s \), \( y_1 < 0.3 \) for \( r_2 = 0 \) and \( r_1 = 0 \) and \( r_2 = 1/s \). We can take \( \lambda_1 = 1 \) and \( \lambda_2 = 0.5 \). The unity feedback loop controller is

\[ C(s) = \frac{1}{s(5s^2 + 34s + 65)} \begin{bmatrix} -(7s^2 + 41s + 34) & -2(s^2 - 8s - 9) \\ 4s^2 + 17s + 17 & -2(3s^2 + 16s + 13) \end{bmatrix} \]

This is in fact a PID controller.
Example 3  Consider the plant described by the following transfer matrix

\[
G(s) = \begin{bmatrix}
\frac{1}{s+1} & s-2 & 0 & 0 \\
0 & 0 & -2e^{-2s} & (s-1)e^{-s}
\end{bmatrix}
\]

The optimal IMC controller is

\[
Q(s) = \begin{bmatrix}
\frac{-(s+1)}{(s+2)(\lambda_1 s+1)} & 0 \\
0 & \frac{-2/5}{\lambda_2 s+1} \\
0 & \frac{-1/5}{\lambda_2 s+1}
\end{bmatrix}
\]

Since the plant is stable, we can implement it in the IMC structure. Suppose the design specification is 10% undershoot with the shortest rise time for both the two loops. We can take lambda1=1.1 and lambda2=0.2.
The result is analytic. This implies researchers can directly compare the proposed method with their own methods or the prevailing methods. Since the controller is analytic, it is easy to verify the robustness.

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