Controller parameterization for SISO and MIMO plants with time delay

Weidong Zhang\textsuperscript{a,b,*}, Frank Allgower\textsuperscript{a}, Tao Liu\textsuperscript{b}

\textsuperscript{a}Institute for System Theory in Engineering, University of Stuttgart, Stuttgart 70550, Germany
\textsuperscript{b}Department of Automation, Shanghai Jiaotong University, Shanghai 200030, PR China

Received 22 July 2004; received in revised form 10 May 2005; accepted 21 March 2006
Available online 8 May 2006

Abstract

Controller parameterization is a very fundamental problem in control theory. It provides an elegant and efficient way towards solving the stabilizing and design problem, with which all stabilizing controllers are characterized and thus a constrained design procedure can be replaced by an unconstrained optimization. In this paper we deal with the problem of characterizing all stabilizing controllers for single-input/single-output (SISO) plants with time delay and multi-input/multi-output (MIMO) plants with multiple time delays. A new parameterization is derived on the basis of the definition of the internal stability. The new parameterization does not depend on the coprime factorization of the plant and has similar form to that of the Youla parameterization for stable plants. An important merit of the proposed parameterization is that it reflects the internal model control (IMC) structure and thus has a very simple relationship to the sensitivity function and complementary sensitivity function. Numerical examples are given to illustrate the proposed parameterization.

© 2006 Elsevier B.V. All rights reserved.

Keywords: Linear system; Stabilization; Parameterization; Asymptotic tracking; SISO; MIMO

1. Introduction

Given a linear time-invariant plant, the design problem can be simply stated as: find a linear time-invariant controller that stabilizes the plant and meets various design objectives, such as the optimal performance criterion. The foundation on which the design procedure is built is the so-called controller parameterization. As we know, the controller design procedure is usually complicated by the fact that only those controllers are allowed for which the closed-loop system is stable. Youla et al.\textsuperscript{[6]} showed that it was possible to parameterize all stabilizing controllers for a particular system in a very effective manner. When designing a controller with specific properties one can simply and without loss of generality search over the space of all stable transfer functions. The parameterization provides a simple means of characterizing all stabilizing controllers for the plant in terms of stable transfer functions and guarantees that the resulting controller automatically yields a closed-loop stable system\textsuperscript{[4]}.

The most well-known form of controller parameterization is the Youla parameterization, which is based on the coprime factorization of the plant. It has been widely used for designing control systems. Morari and Zafiriou\textsuperscript{[4]} provided a parameterization for SISO and MIMO plants, based on which a systematic method was developed for $H_2$ optimal controller design. Dorato and Li\textsuperscript{[1]} proposed an alternative called $U$-parameterization. In the parameterization there is a free design function $U(s)$ that is a bounded real or a strongly bounded real function. The parameterization is applied to various nominal optimization problems including $H_2$ and $H_{\infty}$ optimization and classical design specification optimization. Glaria and Goodwin\textsuperscript{[2]} presented a simple parameterization for the class of all stabilizing controllers for linear minimum phase plants. It has been shown that this result is the dual of the well-known Youla parameterization for the class of all stabilizing controllers for linear stable plants. Zhang et al.\textsuperscript{[8]} gave a modified version which extends the parameterization to any linear plants. Up to now, almost all parameterizations are developed for rational plants without time delay.

\* Corresponding author. Permanent address: Department of Automation, Shanghai Jiaotong University, Shanghai 200030, PR China.

Tel: +86 21 34204019; fax: +86 21 54260762.

E-mail address: wdzhang@sjtu.edu.cn (W. Zhang).
In this paper, we deal with the problem of parameterizing all stabilizing controllers for SISO plants with time delay and MIMO plants with multiple time delays. A new parameterization is derived on the basis of the definition of the internal stability. As a matter of fact, the new parameterization is an extension on that given by Zhang et al. [8]. The new parameterization does not depend on the coprime factorization of the plant and has similar form to that of the Youla parameterization for stable plants. An important merit of the parameterization is that it reflects the IMC structure and has a very simple relationship to the sensitivity function and complementary sensitivity function. This implies that the design procedure of the optimal controller may be significantly simplified.

This paper is organized as follows: In Section 2, the plant under consideration is formally defined and some important background is introduced. In Section 3, the new parameterization is derived for SISO plants with time delay. The result is extended to the MIMO plants with time delay in Section 4. In Section 5 some applications of the new parameterization are discussed. Finally, conclusions are given in Section 6.

2. Preliminaries

Consider the standard feedback configuration shown in Fig. 1, where $G(s)$ is a plant that is linear time invariant and causal, and $C(s)$ is the controller. In general, all signals are assumed to be multivariable and all transfer matrices are assumed to have appropriate dimensions.

A basic requirement for a practice feedback system is the internal stability. This is because all interconnected systems may be unavoidably subject to some nonzero initial conditions and some errors, and it cannot be tolerated in practice that such errors at some locations will lead to unbounded signals at some other locations in the closed-loop system.

The closed-loop system in Fig. 1 is internally stable if bounded signals injected at any point of the control system generate bounded responses at any other point. In a control system many different points can be selected for signal injection and observation, but most of the choices are equivalent for checking internal stability. It is easy to verify that the closed-loop system is internally stable if and only if all elements in the following matrix are stable [4, 8]

$$H(s) = \frac{[G(s)C(s)]-1}{[C(s)I + G(s)C(s)]-1} = \frac{[I + G(s)C(s)]-1G(s)}{-C(s)[I + G(s)C(s)]-1G(s)}.$$  \hspace{1cm} (1)

Two stable matrices $M_r(s)$ and $N_r(s)$ are right coprime if they have the same number of columns and if there exist stable matrices $X_r(s)$ and $Y_r(s)$ such that

$$X_r(s)M_r(s) + Y_r(s)N_r(s) = I.$$  \hspace{1cm} (2)

Similarly, two stable matrices $M_l(s)$ and $N_l(s)$ are left coprime if they have the same number of rows and if there exist stable matrices $X_l(s)$ and $Y_l(s)$ such that

$$M_l(s)X_l(s) + N_l(s)Y_l(s) = I.$$  \hspace{1cm} (3)

A matrix $G(s)$ is said to be double coprime factorization if there exist a right coprime factorization $G(s) = N_lM_l^{-1}$, a left coprime factorization $G(s) = M_r^{-1}N_r$, and stable matrices $X_l(s)$, $Y_l(s)$, $X_r(s)$, and $Y_r(s)$ such that

$$\begin{bmatrix} X_r & Y_r \\ -N_l & M_l \end{bmatrix} \begin{bmatrix} M_r & -Y_l \\ N_l & X_l \end{bmatrix} = I.$$  \hspace{1cm} (4)

Theorem 2.1 (Skogestad and Postlethwaite [5]). Assume that $Y_r(\infty) - \Omega(\infty)N_l(\infty) \neq 0$, the class of all stabilizing controllers can be expressed as

$$C(s) = [Y_l(s) - \Omega(s)N_l(s)]^{-1}[X_l(s) + \Omega(s)M_l(s)],$$  \hspace{1cm} (5)

where $\Omega(s)$ is any stable transfer matrix.

This is the well-known Youla parameterization.

Assume that if the plant is stable, we can take $X_r(s) = X_l(s) = 0$, $Y_r(s) = Y_l(s) = I$, $N_l(s) = N_l(s) = G(s)$, and $M_r(s) = M_l(s) = I$. In this case the Youla parameterization becomes

$$C(s) = (I - \Omega(s)G(s))^{-1}\Omega(s) = \Omega(s)(I - G(s)\Omega(s))^{-1}.$$  \hspace{1cm} (6)

Assume that the model is exact, the sensitivity transfer matrix of the transfer matrix from setpoint $r(s)$ to the error $e(s)$ and the complementary transfer matrix (i.e., the transfer matrix from the setpoint $r(s)$ to the system output $y(s)$) are given by

$$S(s) = [I + G(s)C(s)]^{-1} = [Y_l(s) - N_l(s)\Omega(s)]M_l(s),$$  \hspace{1cm} (7)

$$T(s) = G(s)C(s)[I + G(s)C(s)]^{-1} = N_l(s)[X_l(s) + \Omega(s)M_l(s)].$$  \hspace{1cm} (8)

The most basic objective of a feedback control system is to keep the error between the plant output and the reference small when the overall system is affected by the external disturbance and the uncertainty in the plant. In order to quantify performance, a measure of “smallness” for the error has to be defined. Because the sensitivity function and the complementary sensitivity function reflect the most important relationship of a feedback control system, various performance specifications could be made by weighting the two functions or their combination. For example, the idea of $H_2$ optimal control is to express the control objectives in the form of an $H_2$ optimization problem of the sensitivity function and to determine the controller structure and parameters from its solution.

It is seen that the effect of the unity feedback controller $C(s)$ on the sensitivity function and the complementary sensitivity
Youla parameterization the
Any poles of
Assumption 2.2.
Assumption 2.1.
If
Theorem 2.2
G(s)
ament of
at
is a nongeneric property. Assumption 1(a) is not made for poles
at
s
= 0. Also
G(s)
has a
has no finite zeros on the imaginary axis.

Assumption 2.1. If \( \pi \) is a pole of \( G(s) \) in the open right half plane, then (a) the order of \( \pi \) is equal to 1 and (b) \( G(s) \) has no zeros at \( s = \pi \).

Assumption 1(a) is made solely to simplify the notation. Assumption 1(b) is not very restrictive because the presence of \( z \) zero at \( s = \pi \) implies an exact cancellation in \( \det(G(s)) \), which is a nongeneric property. Assumption 1(a) is not made for poles at \( s = 0 \) because more than one such pole may appear in an element of \( G(s) \), introduced by capacitances present in the plant.

Assumption 2.2. Any poles of \( G(s) \) on the imaginary axis are at \( s = 0 \). Also \( G(s) \) has no finite zeros on the imaginary axis.

These two assumptions allow us to derive the following theorem.

**Theorem 2.2** (Morari and Zafiriou [4]). Assume that Assumptions 1 and 2 hold and that \( Q_0(s) \) stabilizes \( G(s) \), that is, \( H(s) \) is stable for \( Q(s) = Q_0(s) \). Then all \( Q(s) \)'s which make \( H(s) \) stable are given by

\[
Q(s) = Q_0(s) + Q_1(s),
\]

where \( Q_1(s) \) is any stable transfer matrix such that \( G(s)Q_1(s) \) \( G(s) \) is stable.

An important feature of the parameterization is that it preserves the physical meaning of \( Q(s) \) and thus reflects control structure. With this parameterization the controller \( Q(s) \) has a very simple relationship to the sensitivity function and complementary sensitivity function.

\[
S(s) = I - G(s)Q(s),
\]

\[
T(s) = G(s)Q(s).
\]

### 3. Parameterization of SISO plants with time delay

The Youla parameterization cannot be directly used for plants with time delay. In this section, we will develop an alternative controller parameterization for plants with time delay. The new parameterization does not depend on the coprime factorization of the plant and has similar form to that of the Youla parameterization for stable plants. An important merit of the proposed parameterization is that it reflects control structure, like the parameterization in Theorem 2.2.

**Definition 3.1.** A SISO transfer function is proper if it is finite when \( s \to \infty \) and strictly proper if it tends to be zero when \( s \to \infty \). Especially, the plant is semi-proper if it is tends to be a finite nonzero constant when \( s \to \infty \). All transfer functions which are not proper are improper.

**Definition 3.2.** A MIMO transfer function is proper if all its elements are proper and strictly proper if all its elements are strictly proper. Especially, the transfer function is semi-proper if it is proper but not strictly proper. All systems which are not proper are improper.

Consider the unity feedback loop shown in Fig. 1. In general, the plant is described by the transfer function

\[
G(s) = \frac{K N_- (s) N_+ (s)}{M_- (s) M_+ (s)} e^{-\theta s},
\]

where \( K \) is a real constant denoting the static gain, \( \theta \) is a positive real constant denoting the pure time delay. The subscript “-” denotes the roots are in the left half plane and “+” the roots in the closed right half plane, that is, \( N_- (s) \) and \( M_- (s) \) are polynomials with roots in the left half plane, and \( N_+ (s) \) and \( M_+ (s) \) polynomials with roots in the closed right half plane. It is assumed that \( N_- (0) = N_+ (0) = M_- (0) = M_+ (0) = 1 \), and \( \deg(N_- (s)) + \deg(N_+ (s)) \leq \deg(M_- (s)) + \deg(M_+ (s)) \). The first assumption implies that the constant terms of these polynomials are 1 and is made solely to simplify the statement. The second assumption is a standing one, under which the plant is proper.

The plant with imaginary axis poles can be regarded as a special case of the above form with a slight modification: the denominator is the product of a monic polynomial and those imaginary axis poles.
Assume that the plant has \( n \) unstable poles and the unstable pole \( p_j (j = 1, 2, \ldots, r_p) \) is of \( l_j \) multiple, that is,
\[
M_+(s) = \prod_{j=1}^{r_p} (-p_j^{-1}s + 1)^{l_j}.
\]
As
\[
Q(s) = \frac{C(s)}{1 + G(s)C(s)}.
\]
We get
\[
C(s) = \frac{Q(s)}{1 - G(s)Q(s)}.
\]
If the plant is semi-proper, it is assumed that \( 1 - G(s)Q(s) \) is not identically 0. The condition ensures that the transfer matrices from any point of the control system to any other point exist and are proper. Hence, the closed-loop system is internally stable if and only if all elements of the transfer matrix \( H(s) \) are stable:
\[
H(s) = \begin{bmatrix} G(s)Q(s) & [1 - G(s)Q(s)]G(s) \\ Q(s) & -Q(s)G(s) \end{bmatrix}.
\]

**Theorem 3.1.** Assume that \( G(s) \) is a plant with time delay. The unity feedback system in Fig. 1 is internally stable if and only if

1. \( Q(s) \) is stable.
2. \( G(s)[1 - G(s)Q(s)] \) is stable.

Or

1. \( Q(s) \) is stable.
2. \( 1 - G(s)Q(s) \) has zeros wherever \( G(s) \) has unstable poles.
3. There is no right half plane zero-pole cancellation in \( C(s) \).

**Proof.** Necessity is evident. We prove sufficiency. Assume that the two conditions hold, it remains to show that \( G(s)Q(s) \) is stable.

The second condition implies that \( 1 - G(s)Q(s) \) has zeros wherever \( G(s) \) has origin poles. If \( G(s)Q(s) \) is unbounded, \( 1 - G(s)Q(s) \) is unbounded at the same poles. This contradicts the assumption that the second condition holds.

When the plant contains time delay, there may be right half plane zero-pole cancellations in \( C(s) \). Evidently, this should be avoided. \( \square \)

**Example 3.1.** This example is used to illustrate that the third condition is necessary.

Consider the plant with the transfer function
\[
G(s) = \frac{1}{s-1} e^{-0.1s}.
\]
\( G(s) \) has only one right half plane pole at \( s = 1 \). Construct a controller \( C(s) \)
\[
C(s) = \frac{[(4e^{0.1} - 1)s + 1](s - 1)}{(s + 1)^2 - e^{-0.1s}[(4e^{0.1} - 1)s + 1]}.
\]
Notice that the closed-loop system is internally unstable because there exists right half plane zero-pole cancellation at \( s = 1 \) in \( C(s) \). However, all the above conditions are satisfied. The \( Q(s) \) can be written as
\[
Q(s) = \frac{[(4e^{0.1} - 1)s + 1](s - 1)}{(s + 1)^2}.
\]
Obviously, \( Q(s) \) is a stable proper transfer function and has a zero at \( s = 1 \). A little algebra shows that
\[
1 - G(s)Q(s) = 1 - \frac{e^{-0.1s}[(4e^{0.1} - 1)s + 1]}{(s + 1)^2}.
\]
Hence, \( 1 - G(s)Q(s) \) has a zero at \( s = 1 \).

The case with respect to the third condition will only occur in a system where the plant or the controller contains a time delay [7].

**Theorem 3.2.** Assume that \( G(s) \) is a plant with time delay. All controllers that make the unity feedback system internally stable can be parameterized as
\[
C(s) = \frac{Q(s)}{1 - G(s)Q(s)}\frac{Q_1(s)M_+(s)}{K},
\]
where
\[
Q(s) = \frac{Q_1(s)M_+(s)}{K}.
\]
\( Q_1(s) \) is any stable transfer function that makes \( Q(s) \) proper and satisfies
\[
\lim_{s \to p_j} \frac{d^k}{ds^k} \left[ 1 - \frac{Q_1(s)N_+(s)N_-(s)}{M_-(s)} e^{-\theta s} \right] = 0,
\]
\( j = 1, 2, \ldots, r_p, \ 0 \leq k < l_j \)
and there is no right half plane zero-pole cancellation in \( C(s) \).

**Proof.** To guarantee the internal stability of the closed-loop system, first, \( Q(s) \) should be stable.

Second, \( G(s)[1 - G(s)Q(s)] \) should be stable. The implication of this condition is twofold: \( Q(s) \) has to cancel all right half plane poles of \( G(s) \); \( 1 - G(s)Q(s) \) has to cancel all right half plane poles of \( G(s) \). All stable transfer functions that have zeros wherever \( G(s) \) has right half plane poles can be described by
\[
Q(s) = \frac{Q_1(s)M_+(s)}{K},
\]
where \( Q_1(s) \) is a stable transfer function that makes \( Q(s) \) proper. It follows that
\[
1 - G(s)Q(s) = 1 - \frac{Q_1(s)N_+(s)N_-(s)}{M_-(s)} e^{-\theta s}.
\]
That \( 1 - G(s)Q(s) \) has zeros wherever \( G(s) \) has right half plane poles is equivalent to
\[
\lim_{s \to p_j} \frac{d^k}{ds^k} \left[ 1 - \frac{Q_1(s)N_+(s)N_-(s)}{M_-(s)} e^{-\theta s} \right] = 0,
\]
\( j = 1, 2, \ldots, r_p, \ 0 \leq k < l_j \).
Finally, there should be no right half plane zero-pole cancellation in $C(s)$.

It is seen that no coprime factorization is used in the proposed controller parameterization. Instead, one has to test the properness of $Q(s)$ and related constraints. The properness of $Q(s)$ is easy to test. We will show that the related constraints are also easy to test in frequently encountered situations.

**Corollary 3.3.** Assume that $G(s)$ is a plant with time delay and $G(s)$ has only one distinct unstable pole $p_1$. All controllers that make the unity feedback system internally stable can be parameterized as

$$C(s) = \frac{Q(s)}{1 - G(s)Q(s)},$$

where

$$Q(s) = \frac{s - p_1}{K} \left[ \frac{M_-\left(p_1\right)e^{j\theta_p_1} + (s - p_1)Q_2(s)}{N_+\left(p_1\right)N_-\left(p_1\right)} \right],$$

where $Q_2(s)$ is any stable transfer function that makes $Q(s)$ proper and there is no right half plane zero-pole cancellation in $C(s)$.

**Corollary 3.4.** Assume that $G(s)$ is a rational plant. All controllers that make the unity feedback system internally stable can be parameterized as

$$C(s) = \frac{Q(s)}{1 - G(s)Q(s)},$$

where

$$Q(s) = \frac{Q_1(s)M_+(s)}{K},$$

$Q_1(s)$ is any stable transfer function that makes $Q(s)$ proper and satisfies

$$\lim_{s \to p_j} \frac{d}{ds} \left[ 1 - \frac{Q_1(s)N_+(s)N_-(s)}{M_-(s)} \right] = 0, \quad j = 1, 2, \ldots, r_p, \quad 0 \leq k < l_j.$$

It can be verified that the above parameterization is equivalent to that given by Zhang et al. [8]. Thus, the parameterization given by Zhang et al. [8] is a special case of the proposed parameterization.

**Corollary 3.5.** Assume that $G(s)$ is a stable plant. That is, $M_+(s) = 1$. All controllers that make the unity feedback system internally stable can be parameterized as

$$C(s) = \frac{Q(s)}{1 - G(s)Q(s)},$$

where $Q(s)$ is any stable transfer function.

This is the same as the Youla parameterization of stable SISO plants.

**Example 3.2.** Consider a plant with the transfer function

$$G(s) = \frac{s - 2}{(s - 1)(s + 2)}.$$

The plant has only one distinct unstable pole at $s = 1$. According to Corollary 3.3 we have

$$Q(s) = (s - 1)(-3 + (s - 1)Q_2(s)),
$$

where $Q_2(s)$ is any stable transfer function that makes $Q(s)$ proper. All controllers that make the unity feedback system internally stable can be parameterized as

$$C(s) = \frac{(s - 1)(s + 2)[-3 + (s - 1)Q_2(s)]}{(s + 2) - (s - 2)[-3 + (s - 1)Q_2(s)]}.$$

**Example 3.3.** Consider the problem of stabilizing the plant

$$G(s) = \frac{1}{(s - 1)(s - 2)}.$$

The plant has two distinct unstable poles at $s = 1$ and $s = 2$. Then

$$Q(s) = (s - 1)(s - 2)Q_1(s),$$

where $Q_1(s)$ is a stable transfer function satisfying

$$\lim_{s \to 1} [1 - Q_1(s)] = 0,$$

$$\lim_{s \to 2} [1 - Q_1(s)] = 0.$$

So that

$$Q_1(s) = 1 + (s - 1)(s - 2)Q_2(s),$$

where $Q_2(s)$ is any stable transfer function that makes $Q(s)$ proper. All controllers that make the unity feedback system internally stable can be parameterized as

$$C(s) = \frac{1 + (s - 1)(s - 2)Q_2(s)}{-Q_2(s)} = (s - 1)(s - 2) - \frac{1}{Q_2(s)}.$$

**4. Parameterization of MIMO plants with time delay**

In this section we will extend the controller parameterization of SISO plants with time delay to MIMO plants with time delay. Assume that the plant in Fig. 1 is of dimension $p \times q$ and satisfies that

1. There is no right half plane zero-pole cancellation in $G(s)$.
2. $\det[I - G(s)Q(s)]$ is not identically zero if the plant is semi-proper. The condition ensures that the transfer matrices from any point of the control system to any other point exist and are proper.

These assumptions are not very restrictive because they will not be destroyed after a slight perturbation in the coefficients of the plant is introduced.
As
\[ C(s) = Q(s)[I - G(s)Q(s)]^{-1}. \]
The closed-loop system is internally stable if and only if all elements of the following transfer matrix are stable:
\[ H(s) = \begin{bmatrix} G(s)Q(s) & I - G(s)Q(s)G(s) \\ Q(s) & -Q(s)G(s) \end{bmatrix}. \] 

**Theorem 4.1.** The unity feedback control system is internally stable if and only if
\[ Q(s) \text{ is stable.} \]
\[ I - G(s)Q(s)G(s) \text{ is stable.} \]

**Proof.** Assume that \([I - G(s)Q(s)]G(s)\) is stable but \(G(s)Q(s)\) and \(Q(s)G(s)\) are not stable. Then \(G(s)Q(s)\) is unbounded at the unstable poles of \(G(s)\). If \(G(s)Q(s)\) is unbounded, \([I - G(s)Q(s)]G(s)\) is unbounded at the unstable poles of \(G(s)\). This contradicts the assumption. Thus, the condition \([I - G(s)Q(s)]G(s)\) is stable implies that \(G(s)Q(s)\) is stable. As
\[ [I - G(s)Q(s)]G(s) = G(s)[I - Q(s)G(s)]. \]
The condition \([I - G(s)Q(s)]G(s)\) is stable also implies that \(Q(s)G(s)\) is stable. □

**Corollary 4.2.** The unity feedback control system is internally stable if and only if
\[ (1) \quad Q(s) \text{ is stable.} \]
\[ (2) \quad I - G(s)Q(s)G(s) \text{ is stable.} \]
\[ (3) \quad \text{There is no right half plane zero-pole cancellation in } [I - G(s)Q(s)]G(s) \text{ and } C(s). \]

**Proof.** The condition that \([I - G(s)Q(s)]G(s)\) is stable implies that \(I - G(s)Q(s)\) has to cancel all right half plane poles of \(G(s)\), or equivalently, \(I - G(s)Q(s)\) has to have zeros wherever \(G(s)\) has unstable poles. However, for multivariable systems not all \(Q(s)\) satisfying this can guarantee that \([I - G(s)Q(s)]G(s)\) is stable. There exist such possibilities that \(I - G(s)Q(s)\) has zeros wherever \(G(s)\) has unstable poles, but there are right half plane zero-pole cancellations that cannot be removed in \([I - G(s)Q(s)]G(s)\) and \(C(s)\). Thus, the condition \(I - G(s)Q(s)\) has zeros wherever \(G(s)\) has unstable poles is only necessary for internal stability. Only those \(Q(s)\)s that make right half plane zero-pole cancellations in \([I - G(s)Q(s)]G(s)\) and \(C(s)\) removable can guarantee the stability of \([I - G(s)Q(s)]G(s)\). □

**Example 4.1.** This example is used to illustrate irremovable right half plane zero-pole cancellations in \([I - G(s)Q(s)]G(s)\). The plant is described by the transfer matrix
\[ G(s) = \begin{bmatrix} 1 \\ s + 3 \\ s + 2 \end{bmatrix}. \]
It has two poles at \(s = -3\) and \(s = 2\) and one nonnude zero at \(s = 3\). Assume that we take the controller
\[ Q(s) = \begin{bmatrix} -s - 1 \\ s + 1 \\ s + 2 \end{bmatrix}. \]
\(Q(s)\) is stable. Since
\[ I - G(s)Q(s) = \begin{bmatrix} s(s + 5) & 0 \\ (s + 3)(s + 1) & s(s + 29)(s - 2) \end{bmatrix}. \]
\(I - G(s)Q(s)\) has zeros wherever \(G(s)\) has unstable poles. However,
\[ [I - G(s)Q(s)]G(s) = \begin{bmatrix} \frac{s(s + 5)}{(s + 3)^2(s + 1)} & \frac{s(s + 5)}{2s(s + 29)(s - 2)} \end{bmatrix}. \]
which is not stable. It has a right half plane pole at \(s = 2\) and a right half plane zero \(s = 2\), but the right half plane zero-pole cancellation cannot be removed.

Assume that \(G(s)\) has \(r_p\) unstable poles and the unstable pole \(p_j (j = 1, 2, \ldots, r_p)\) is of \(l_j\) multiple.

**Theorem 4.3.** Assume that \(G(s)\) is a plant with time delay. All controllers that make the unity feedback system internally stable can be parameterized as
\[ C(s) = Q(s)[I - G(s)Q(s)]^{-1}, \]
where \(Q(s)\) is a stable proper transfer matrix that satisfies
\[ \lim_{s \to p_j} \frac{d^k}{ds^k} \det[I - G(s)Q(s)] = 0, \]
\[ j = 1, 2, \ldots, r_p, \quad 0 \leq k < l_j \]
and there is no right half plane zero-pole cancellation in \([I - G(s)Q(s)]G(s)\) and \(C(s)\).

**Proof.** To guarantee the internal stability of the closed-loop system, first, \(Q(s)\) should be stable. This implies that \(Q(s)\) should be proper.

Second, \([I - G(s)Q(s)]G(s)\) should be stable. Thus, \(1 - G(s)Q(s)\) has to cancel all right half plane poles of \(G(s)\). To achieve this \(Q(s)\) has to satisfy that
\[ \lim_{s \to p_j} \frac{d^k}{ds^k} \det[I - G(s)Q(s)] = 0, \]
\[ j = 1, 2, \ldots, r_p, \quad 0 \leq k < l_j \]
The condition cannot guarantee \([I - G(s)Q(s)]G(s)\) stable unless there is no right half plane zero-pole cancellation in \([I - G(s)Q(s)]G(s)\) and \(C(s)\). □
Corollary 4.4. Assume that $G(s)$ is a plant with time delay and $g(s)$ has only one distinct unstable pole $p_1$. All controllers that make the unity feedback system internally stable can be parameterized as

$$C(s) = Q(s)[I - G(s)Q(s)]^{-1},$$

where $Q(s)$ is any stable proper transfer matrix that satisfies

$$\det[I - G(p_1)Q(p_1)] = 0$$

and there is no right half plane zero-pole cancellation in $[I - G(s)Q(s)]G(s)$ and $C(s)$.

Corollary 4.5. Assume that $G(s)$ is a stable plant. All controllers that make the unity feedback system internally stable can be parameterized as

$$C(s) = Q(s)[I - G(s)Q(s)]^{-1},$$

where $Q(s)$ is any stable proper transfer matrix.

It should be pointed out that the proposed approach is general enough for more general linear plants, for example, the plant with state delay. In this case the plant contains multiple time delays.

The state time delay is seldom considered in frequency domain methods, because the plant with state delay usually contains infinity number of unstable poles. A plant with output and state delays can be described by the state equations of the form

$$\begin{align*}
\dot{x}(t) &= A_0x(t) + \sum_{i=1}^{n_1} A_i x(t - \vartheta_i) + B_0u(t) + \sum_{j=1}^{n_j} B_j u(t - \tau_j), \\
y(t) &= C_0x(t) + \sum_{r=1}^{n_r} C_r x(t - \zeta_r),
\end{align*}$$

(15)

where $\vartheta_i$, $\tau_j$, and $\zeta_r$ are lumped delays. The corresponding transfer matrix can then be written as

$$G(s) = \left( C_0 + \sum_{r=1}^{n_r} C_r e^{-\zeta_rs} \right) \left( sI - A_0 - \sum_{i=1}^{n_1} A_i e^{-\vartheta_is} \right)^{-1}$$

$$\times \left( B_0 + \sum_{j=1}^{n_j} B_j e^{-\tau_js} \right).$$

(16)

We can rewrite it into the form of

$$G(s) = \begin{bmatrix}
G_{11}(s)e^{-\vartheta_1s} & \cdots & G_{1n}(s)e^{-\vartheta_ns} \\
\vdots & \ddots & \vdots \\
G_{n1}(s)e^{-\vartheta_1s} & \cdots & G_{nn}(s)e^{-\vartheta_ns}
\end{bmatrix}.$$
where $Q_1(s)$ is stable. If
\[
\lim_{s \to 0} 1 - G(s)Q(s) = 0.
\]
The closed-loop system possesses the asymptotic property. This implies that
\[
Q_1(s) = 1 + sQ_2(s),
\]
where $Q_1(s)$ is stable. This leads to
\[
1 - G(s)Q(s) = 1 - \frac{[1 + sQ_2(s)]N_+(s)N_-(s)e^{-\theta s}}{M_-(s)}.
\]
Thus, $Q_2(s)$ should satisfy
\[
\lim_{s \to p_j} \frac{d^k}{ds^k} \left\{ 1 - \frac{[1 + sQ_2(s)]N_+(s)N_-(s)e^{-\theta s}}{M_-(s)} \right\} = 0, \quad j = 1, 2, \ldots, r_p, \quad 0 \leq k < l_j.
\]

Now consider the MIMO plant $G(s)$. Assume that $G(s)$ has normal row rank. Otherwise, $T(s)$ must be identically singular. The right inverse of $G(s)$ is $G^+(s)$. Let $m$ be the largest integer for which
\[
\text{rank}[\lim_{s \to 0} s^m L(s)] = p.
\]
Then the system $L(s)$ is said to be of Type $m$. Note that $L(s)$ has at least $p \times m$ poles at the origin. For a Type $m$ system, the sensitivity transfer function satisfies
\[
\lim_{s \to 0} s^{-k} S(s) = 0, \quad 0 \leq k < m
\]

If the closed-loop system is stable, as $t \to \infty$, the system can perfectly track inputs of the form $\sum_{k=0}^{m} a_k s^{-k}$, where $a_k$ are real constant vectors. In particular, a Type 1 system requires that
\[
\lim_{s \to 0} G(s)Q(s) = I.
\]

**Theorem 5.2.** Assume that $G(s)$ is a plant with time delay. All controllers that make the unity feedback system internally stable and have a zero steady-state error for a step reference can be parameterized as
\[
C(s) = Q(s)[I - G(s)Q(s)]^{-1},
\]
where
\[
Q(s) = G^+(0)[I + sQ_1(s)].
\]
$Q_1(s)$ is any stable transfer matrix that makes $Q(s)$ proper and satisfies
\[
\lim_{s \to p_j} \frac{d^k}{ds^k} \det[I - G(s)G^+(0) - sG(s)G^+(0)Q_1(s)] = 0, \quad j = 1, 2, \ldots, r_p, \quad 0 \leq k < l_j.
\]
And there is no right half plane zero-pole cancellation in $[I - G(s)Q(s)]G(s)$ and $C(s)$.

**Proof.** If
\[
\lim_{s \to 0} [I - G(s)Q(s)] = 0.
\]
The closed-loop system possesses the asymptotic property. Then
\[
Q(0) = G^+(0).
\]
All transfer matrices that satisfy the condition can be written as
\[
Q(s) = G^+(0)[I + sQ_1(s)].
\]
Then the result follows from applying Theorem 4.1. □

Often it is desired in practice that the closed-loop response is decoupled, that is, the closed-loop transfer matrix is diagonal. Gundes [3] gave a decoupling parameterization for rational plants. Based on the proposed parameterization, we can develop an alternative. The decoupled closed-loop transfer matrix can be written as
\[
T(s) = G(s)Q(s) = G(s)G^+(0)[I + sQ_1(s)].
\]
Let $m_{ij}$ be the largest multiplicity of $p_j$ in any element of the $i$th row of $G(s)$ and the $i$th element of $T(s)$ is $T_i(s)$.

**Corollary 5.3.** Assume that the closed-loop response is decoupled. All controllers that make the unity feedback system internally stable and have a zero steady-state error for a step reference can be parameterized as
\[
C(s) = Q(s)[I - G(s)Q(s)]^{-1},
\]
where
\[
Q(s) = G^+(0)[I + sQ_1(s)].
\]
$Q_1(s)$ is any stable transfer matrix that makes $Q(s)$ proper and $T(s)$ diagonal, and satisfies
\[
\lim_{s \to p_j} \frac{d^k}{ds^k} [1 - T_i(s)] = 0, \quad j = 1, 2, \ldots, r_p, \quad 0 \leq k < m_{ij}.
\]
And there is no right half plane zero-pole cancellation in $[I - G(s)Q(s)]G(s)$ and $C(s)$.

Corollary 5.3 implies that more right half plane zeros in $I - G(s)Q(s)$ has to be introduced to guarantee the internal stability. A question of interest is when we have to do this. We have to do this if the multiples of the right half plane poles of at least one row of $G(s)$ are not the same. This can be illustrated by the following example.

**Example 5.1.** Consider the following plant
\[
G(s) = \begin{bmatrix}
1 & 2 \\
1 & 1 \\
\frac{1}{s-2} & \frac{1}{s-2}
\end{bmatrix},
\]
which has two poles at \( s = -3 \) and \( s = 2 \) and one zero at \( s = 3 \).

If we take the following controller:

\[
Q(s) = \begin{bmatrix}
-\frac{(s - 1)}{s + 1} & \frac{2(s - 2)(22s + 1)}{(s + 3)(s + 1)^2} \\
\frac{1}{s + 1} & -\frac{(s - 2)(22s + 1)}{(s + 3)(s + 1)^2}
\end{bmatrix}
\]

The closed-loop response is decoupled. It is evident that both

\[
I - G(s)Q(s) = \begin{bmatrix}
\frac{s(s + 5)}{(s + 3)(s + 1)} & 0 \\
0 & \frac{s(s + 29)(s - 2)}{(s + 3)(s + 1)^2}
\end{bmatrix}
\]

and

\[
[I - G(s)Q(s)]G(s) = \begin{bmatrix}
\frac{s(s + 5)}{(s + 3)^2(s + 1)} & \frac{2s(s + 5)}{(s + 3)^2(s + 1)} \\
\frac{(s + 29)(s - 2)}{(s + 3)(s + 1)^2} & \frac{s(s + 29)(s - 1)}{(s + 3)(s + 1)^2}
\end{bmatrix}
\]

are stable and only one right half plane zero is introduced in \( I - G(s)Q(s) \).

6. Conclusions

A fundamental requirement of any feedback control system is the stability of the closed-loop system. The design problem may therefore be considered as a search or optimization over the class of all stabilizing controllers. Controller parameterization provides an elegant and efficient way for the design problem, with which a constrained search or optimization can be replaced with an unconstrained one. Therefore, controller parameterization opens the door to various optimization-based design strategies.

In this paper we derived a new parameterization. Some applications of the proposed parameterization are given. The parameterization is a natural extension to that given by Zhang et al. [8]. In comparison with existing parameterizations, features of the proposed parameterization are that it is applicable to nonsquare plants with time delays, and can provide decoupling response and asymptotically tracking response. Our current work is to develop new design methods for plants with time delay based on the proposed parameterization.

Acknowledgment

This work was finished when Prof. Weidong Zhang was a Humboldt Research Fellow at the IST of the University of Stuttgart with Prof. Dr.-Ing F. Allgower. We are thankful to the anonymous reviewer for his/her valuable suggestions on improving the paper. This paper is supported by National Science Foundation of China (60474031), NCET (04-0383), SRSP (04QMH1405), and the Alexander von Humboldt Research Fellowship.

References


